Fractals in the Classroom

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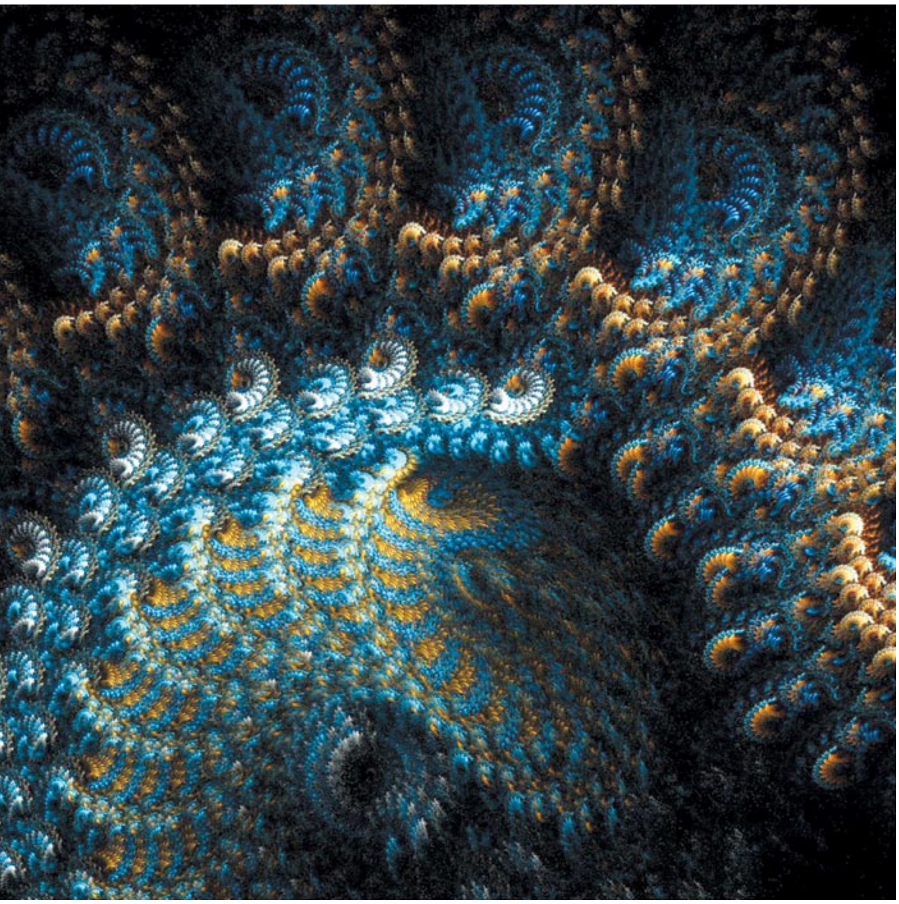

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What exactly is a fractal?

Traditionally, students learn about the familiar forms of symmetry: reflection, translation, and rotation. Intuitively, fractals are symmetric with respect to magnification. A magnification of a small part of the fractal looks essentially the same as the entire picture. More formally, fractals have the property of self-similarity—that is, a fractal is any shape that is made up of smaller copies of itself. Self-similarity is what distinguishes fractals from most conventional Euclidean figures and makes them appealing. Do fractals hold the same characteristics as other Euclidean objects? Fractals offer much to explore for even very young students.

In the past, we have taught fractal geometry to students of various ages and abilities, introducing the idea of self-similarity to kindergarten and middle school children through workshops tailored to each of these age groups. Further, we have led in-depth investigations with nonmajors in liberal arts courses and have gone into even more detail with mathematics majors in upper-level geometry courses. Fractal geometry offers teachers great flexibility: It can be adapted to the level of the audience or to time constraints. Although easily explained, fractal geometry leads to rich and interesting mathematical complexities.

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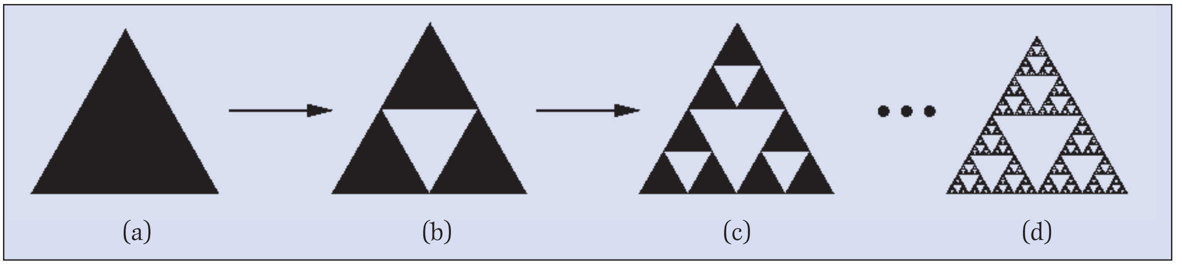

Fig. 1 The Sierpinski triangle can be constructed by connecting the midpoints of each side of a filled-in triangle and removing the resulting down-pointing triangle. The process can be iterated.

At the beginning of our workshops, we share with students the two reasons we love geometry—because it is so old and because it is so new. Euclid's work has endured for more than two thousand years. Fractal geometry, in contrast, is a relatively new area of mathematics, first formalized in the 1960s, and many mathematicians today work in this field. Because fractals are so recent and different, using them to present even old ideas can breathe new life into a classroom. Traditional topics such as measurement can seem fresh and exciting when applied to fractals. Fractal geometry gives students a new perspective on their mathematical understanding and encourages creativity in their problem solving.

Most human-made structures are Euclidean; they have nice straight lines or smooth curves. Nature does not follow the same rules

Another motivation for studying fractal geometry comes from observation of natural phenomena. Most human-made structures are Euclidean; they have nice straight lines or smooth curves. Nature does not follow the same rules. Natural objects have many crevices, wiggles, and jagged edges—all fractal characteristics. The details of nature cannot be measured truly by using only Euclidean tools. For instance, if you measured the outline of a tree, a cloud, or a coastline by using a Euclidean (straight) ruler, you would obtain different measurements depending on the ruler's scale. The question is not "What did you do wrong?" but rather "Did you use the wrong tool?" Marking off the outline by using a stick that is one foot long will give a result different from that obtained by using a stick that is one inch long. The smaller stick captures more of the object's detail. Thus, in many ways, fractal geometry is a better tool for modeling nature.

AN APPLICATION

The main tool for creating a fractal is iteration: One applies a process, takes the result, and continues to reapply the process to obtain a mathematical object with striking characteristics. Suppose one starts with a filled-in equilateral triangle. Connect the midpoints of the three sides and remove the small inverted triangle just created. Three smaller filled-in equilateral triangles are left. Now, iterate this process—that is, apply it to each of the smaller triangles. Repeat the iteration with the nine smaller triangles that result and then again with the twenty-seven still smaller ones. The limiting shape of this process, the shape that results from infinite repetition, is a fractal known as the Sierpinski triangle (see **figs. 1a–d**). Note the self-similarity that results: The shape is made of scaled copies of itself.

Constructing fractals by using iteration can appeal to a wide range of students and can be introduced in a short time. The students are required only to master a relatively simple process (e.g., connecting midpoints and removing a triangle), but when they apply this with a little patience they are rewarded with a very intricate picture.

Once students have constructed a fractal such as the Sierpinski triangle, the teacher can ask several questions about it. For instance, what is the triangle's area? This question can be answered in various ways depending on the level of the class. First, have students assume that the area of the original triangle is 1. Then have them make a table listing the total area of the figure at each stage in the construction (see **table 1**).

As students compare the areas at each stage, they should notice that a pattern emerges. The decimals indicate that the areas form a sequence that decreases to zero. The fractions indicate the pattern of powers showing that the area tends to zero. In precalculus classes, students can follow the pattern to its logical conclusion. In calculus classes, this example lends itself well to an application of the limit process. With further thought, most students can find an explanation for why the pattern must hold.

Students could compute the remaining area in the figure by considering how much of the original area is removed in the construction process. In the first stage, 1/4 was removed; in the second stage, $3(1/4)^2$ was removed; and so on. In the nth stage, $3^{n-1}(1/4)^n$ was removed. Now, to find the total area removed, simply add all these fractions. In other words, evaluate the infinite sum

$$\sum_{n=3}^{\infty} 3^{n-1}\left(\frac{1}{4}\right)^{n} = 1.$$

This example of a geometric series demonstrates that the Sierpinski triangle has zero area.

In either case, the answer seems suprising at first. If the Sierpinski triangle is an object with zero area, does that mean all its points have been removed and nothing remains? But certainly the edges of the original triangle are never removed. After all, only the middle triangles have been removed. By the same logic, the edges of the stage-one triangles must remain in the fractal. This leads to another question: What is the perimeter of the Sierpinski triangle? This question too can be answered by setting up a table for patterns, applying logic to explain the patterns, and using a variety of tools.

Table 1

Areas of Successive Stages in the Sierpinski Triangle (the area of the original triangle is assumed to be 1)

Stage	Number of Triangles	Area of Each Triangle	Total Area at this Stage	
0	1	1	1	
1	3	$\frac{1}{4}$	$\frac{3}{4} = 0.75$	
2	9	$\frac{1}{16}$	$\frac{9}{16} \approx 0.563$	
3	27	$\frac{1}{64}$	$\frac{27}{64} \approx 0.422$	
n	3^n	$\frac{1}{4^n}$	$\left(\frac{3}{4}\right)^n$	

CONCLUSION

In this article, we have described fractal geometry and provided a sample exercise. However, the Sierpinski triangle is by no means the only fractal that is useful in the classroom; many more fractals and their applications can be constructed. Some are as easy to form as the Sierpinski triangle, while others can be as complicated as the Mandelbrot set, which is constructed by iterating a function on the complex plane. Through fractal geometry, students will investigate a range of topics, including sequences, symmetry, ratio and proportion, measurement, and fractions. At a higher level, tools such as logarithms, the composition of functions, Pascal's triangle, arithmetic in different bases, and complex numbers can be applied.

Fractal activities can be found that address most NCTM Standards. Thus, fractals can be taught separately or incorporated as examples into traditional lessons. For additional ideas and detailed lesson plans, refer to the *Tool Kit of Dynamics Activities* collection published by Key Curriculum Press (see Choate and Devaney listings) and the electronic book *Fractal Geometry* (Frame, Mandelbrot, and Neger 2006). We hope you will consider fractal geometry as a resource that reinforces the concepts in your current curriculum and introduces your students to a new and beautiful field of mathematics.

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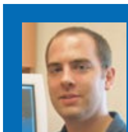

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